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Bounded asymptotic solutions for incomplete contacts in partial slip

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Abstract

Asymptotic solutions for the contact pressure and shearing tractions present adjacent to the edge of an incomplete contact are derived, and for each a dimensional scaling factor (akin to a generalised stress intensity factor), is defined. These two quantities may therefore be used to define the complete local stress state, for which purpose the Muskhelishvili potentials are derived. The application of this technique to a comparison of the Hertz–Cattaneo contact, the flat and rounded contact and the tiled punch is described, and the extent of the domain of validity of the asymptote found.

1. Introduction

We have recently developed the idea of representing the state of stress in the neighbourhood of the edge of a slipping complete contact by using an asymptotic solution (Mugadu et al., 2002). This is a convenient way of encapsulating, in a single parameter, all of the features of the local state of stress, giving a simple way of comparing the fatigue strength measured in a test, and using it to predict the strength of a complex prototype. Of course, it is only the very local state of stress which may be represented in this way, but this is precisely what is required when the process region in which cracks nucleate is to be quantified and characterised. It is necessary only for the process zone itself (which might crudely be thought of as being the region in which irreversibilities arise) to be completely surrounded by a zone in which the asymptotic expression swamps higher order terms. Note that, in a real problem, macroscopic plasticity is rather unlikely, but some local measure of grain stress-localisation, as postulated by Dang Van et al. (1989), simply means that the fatigue limit, as measured in a plain fatigue test, may be used to represent effective yield strength. In the same vein, the idea of a more complex asymptote, incorporating very local rounding at the edge of what is otherwise a complete contact, was introduced (Sackfield et al., 2003), and, in this paper, we turn our attention to the development of an asymptotic correlation procedure for *incomplete* contacts.

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At first sight, it might be thought that the asymptotic approach, as classically defined as a power series expansion in the local radial coordinate of the state of stress, was a poor candidate for incomplete contacts, as the magnitude of the stress state, at least as measured by von Mises' parameter, becomes more severe as one moves away from the contact edge. This is certainly so if the contact is sliding, but here we are concerned with fretting, so that the contact is almost invariably in partial slip, and the stick/slip interface forms a natural barrier to the region where cracks may form. It follows that we require an asymptote which satisfactorily describes the state of stress only within the slip region and, possibly, a small adjacent stick region. However, there are additional requirements, here, over and above those needed for a singular asymptote, viz. a means of describing the stick–slip regime and extent through a simple scaling factor.

The aim is to provide a simple description of the local state of stress with as few parameters as possible: it will transpire that, if there is no remote bulk tension present, two will be needed: one serves as a scaling factor for the influence of the normal contact force, and will be denoted K_{I}. It will have dimensions of $[FL^{-5/2}]$. The second serves as a scaling factor for the shearing force, will be denoted by K_{II}, and will have dimensions $[FL^{-3/2}]$. These quantities perform several functions; they will quantify the local distribution of stress, they will define the partial slip region, and they will implicitly specify the *scale* of the contact.

We will first derive the asymptote itself, and its characteristics will be explored. It will then be applied to example closed form problems (the Hertzian contact, the flat and rounded contact, and the tilted punch) as demonstrations of how the scaling factors K_{I}, K_{II} may be derived, and also of the quality of the state of stress implied by the asymptote. It should be noted, however, that the intended application of this procedure extends beyond a comparison of simple closed-form contacts: the intention is to provide a tool for matching the state of stress in any incomplete contact, including one solved numerically, normally by the finite element method; for example the typical fan blabe dovetail.

2. Formulation

Consider a conventional Hertzian contact, as displayed in Fig. 1(a). From standard results (Hills et al., 1993), the contact pressure distribution may be written down as

$$p(x) = p_0\sqrt{1-\left(\frac{x}{a}\right)^2}, \tag{1}$$

where p_0 is the peak contact pressure. Move the origin to the left hand edge by the change of coordinate

$$x = (t-a) \tag{2}$$

so that, if we expand the resulting expression by the binomial theorem, we find that

$$p(t) \simeq p_0\sqrt{2\frac{t}{a}} \quad 0 < \frac{t}{a} \ll 1. \tag{3}$$

This suggests that the contact pressure adjacent to the edge of this and other incomplete contacts may be written as (Fig. 1(b))

$$\begin{aligned} p(t) &= K_{\mathrm{I}}\sqrt{t} \quad t > 0 \qquad K_{\mathrm{I}} : [FL^{-5/2}] \\ &= 0 \quad t < 0 \end{aligned} \tag{4}$$

which is consistent with the formulation of any incomplete half-plane problem, when the Riemann–Hilbert procedure is invoked—it corresponds to the form of the fundamental function. We shall also require the corresponding Muskhelishvili potential, in order to determine the complete internal state of stress. In this

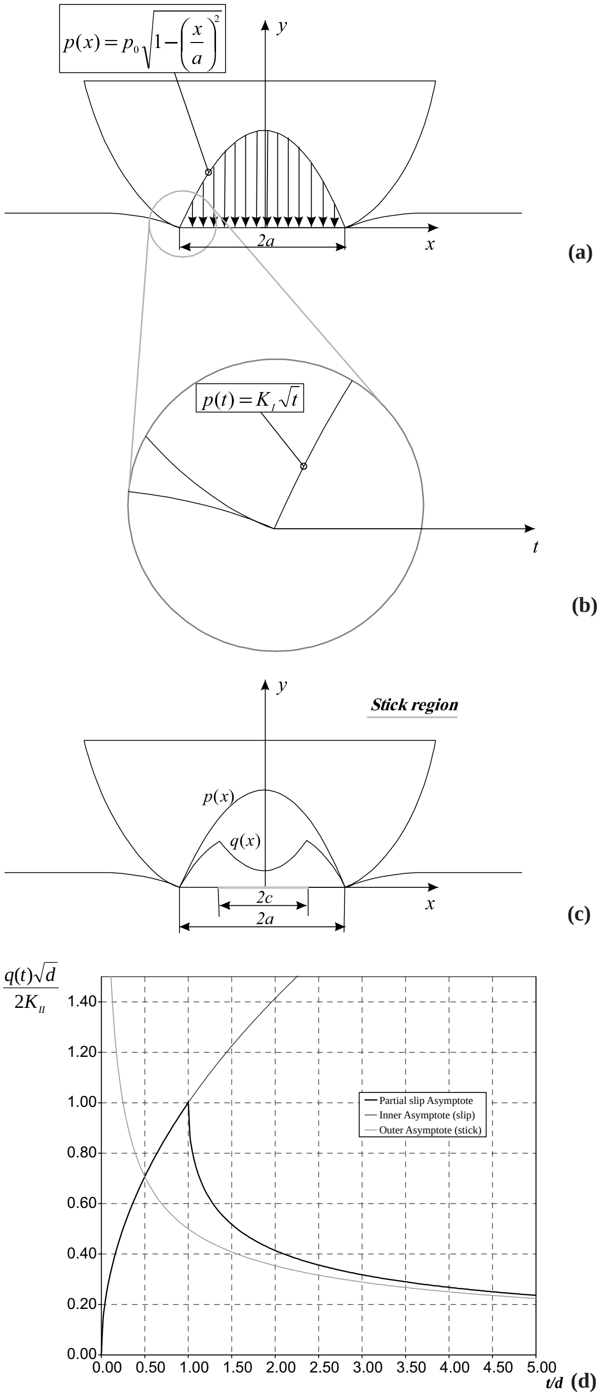

Fig. 1. Hertzian contact: (a) conventional contact configuration, (b) asymptotic characteristic and (c) partial-slip configuration. (d) Partial-slip shear asymptote: solution with inner and outer characteristics.

instance, the best way to derive the solution is to take the potential for a finite Herztian contact (Eq. (1)), given by

$$\Phi(z) = \frac{\mathrm{i}p_0}{2a}[z - \sqrt{z^2 - a^2}], \tag{5}$$

where $z = x + \mathrm{i}y$, $\mathrm{i} = \sqrt{-1}$. Again move the origin to the edge of the contact by dint of the transformation

$$z = w - a \tag{6}$$

so that (Dini et al., 2004)

$$\begin{aligned}\Phi(w) &= \frac{p_0}{2a}(\mathrm{i} + f)[w - a - \mathrm{i}\sqrt{2aw}\sqrt{1 - w/2a}] \\ &= \frac{p_0}{2a}(\mathrm{i} + f)[w - a - \mathrm{i}\sqrt{2bw}\{1 - w/4a + \ldots\}], \quad |w| \ll 4a.\end{aligned} \tag{7}$$

Therefore, for small w, and ignoring the constant term $\frac{p_0}{2a}(\mathrm{i} + f)[-a]$ in $\Phi(w)$ (as this simply produces rigid body movement and does not affect the stresses), we see that, adjacent to the contact edge:

$$\Phi(w) \simeq \frac{p_0}{2a}(\mathrm{i} + f)[-\mathrm{i}\sqrt{2aw}] = \frac{p_0}{\sqrt{2a}}\sqrt{w} \quad |w| \ll a. \tag{8}$$

Lastly, use the definition of K_I to re-scale the solution (i.e. replace p_0), so that the potential corresponding to contact pressure (Eq. (4)), is given by

$$\Phi(w) = \frac{K_\mathrm{I}}{2}\sqrt{w} \quad \forall w. \tag{9}$$

Note that the solution just derived may be collocated with that for any finite incomplete contact at the extreme edge of the contact, i.e. as $t \to 0$, Eq. (4). We now turn our attention to the effects of applying a shearing force.

2.1. Shearing force

A means is sought of characterising the influence of a shearing force less than that needed to cause sliding. The intention is, of course, to find a self-contained partial-slip solution within a semi-infinite contact formulation. Thus, the size of the slip zone becomes the only length dimension present in the solution. The starting point is the standard partial-slip finite Hertzian contact, associated with the names Cattaneo (1938) and Mindlin (1949), and which has the configuration shown in Fig. 1(c). The shearing traction distribution is given by

$$\begin{aligned}q(x) &= fp(x) \quad -a < x < -c \cap c < x < a \\ &= fp(x) - fp_0\left(\frac{c}{a}\right)\sqrt{1 - \left(\frac{x}{c}\right)^2} \quad |x| \leqslant c,\end{aligned} \tag{10}$$

where c is the extent of the stick zone. As before, make a transformation of coordinate to the contact edge by setting $x = t - a$, and introduce an arbitrary scaling factor for the magnitude of the solution, Λ, to give

$$\begin{aligned}q(t) &= \Lambda\sqrt{\frac{a}{2}}\sqrt{2\frac{t}{a} - \left(\frac{t}{a}\right)^2} \quad 0 < t < d \\ &= \Lambda\sqrt{\frac{a}{2}}\left(\sqrt{2\frac{t}{a} - \left(\frac{t}{a}\right)^2} - \left(\frac{a-d}{a}\right)\sqrt{1 - \left(\frac{t-a}{a-d}\right)^2}\right) \quad t > d\end{aligned} \tag{11}$$

with $d = c - a$. Now make the contact semi-infinite in extent by letting $a \to \infty$, whilst keeping d finite. This results in

$$\Lambda \lim_{a\to\infty} \sqrt{\frac{a}{2}}\left(\sqrt{2\frac{t}{a} - \left(\frac{t}{a}\right)^2} - \left(\frac{a-d}{a}\right)\sqrt{1 - \left(\frac{t-a}{a-d}\right)^2}\right) = \Lambda(\sqrt{t} - \sqrt{t-d}). \tag{12}$$

We now introduce the definition of the shearing mode scaling factor, K_{II}. We know that, if $t \gg d$ the effect of the slip zone will become very small, and so we expect the shear traction, $q(t)$, to assume a form consistent with a constant surface tangential displacement, i.e. $q(t) \sim 1/\sqrt{t}$. We may therefore define

$$q(t) = K_{\mathrm{II}}\frac{1}{\sqrt{t}} \quad t \gg d \qquad K_{\mathrm{II}} : [FL^{-3/2}] \tag{13}$$

but we also know that the general form of the solution will look like (12) for values of $t > d$. Therefore, from (12), we can write:

$$\begin{aligned} q(t) &= \Lambda\sqrt{t} \quad 0 < t < d \\ &= \Lambda(\sqrt{t} - \sqrt{t-d}) \quad t > d && (14) \\ &= \Lambda\frac{d}{2\sqrt{t}} \quad t \gg d && (15) \end{aligned}$$

with Λ to be found. Equation (15) is derived from Eq. (14) by considering $t \to \infty$. By comparing Eqs. (13) and (15), we see that $\Lambda = 2K_{\mathrm{II}}/d$ so that Eq. (13) applies and, in addition

$$\begin{aligned} q(t) &= \frac{2K_{\mathrm{II}}}{d}\sqrt{t} \quad 0 < t < d \\ &= \frac{2K_{\mathrm{II}}}{d}(\sqrt{t} - \sqrt{t-d}) \quad t > d \\ &= 0 \quad t < 0. \end{aligned} \tag{16}$$

Furthermore, we can also write

$$q(t) = fK_{\mathrm{I}}\sqrt{t} \quad 0 < t < d \tag{17}$$

so that, by equating the state of shear in the slip region, the scaling factors K_{I}, K_{II} are seen to be related by

$$\frac{K_{\mathrm{II}}}{K_{\mathrm{I}}} = \frac{fd}{2}. \tag{18}$$

Fig. 1(d) shows the shearing asymptote together with its inner and outer characteristics.

These results imply that by assigning K_{I} and K_{II} or, equivalently, K_{I} and d, from any given finite incomplete contact suffering partial slip, and remote from the sliding condition, a full description of the characteristics of the contact in the neighbourhood of the contact edge can be obtained in closed form using the asymptote. It should be noted, though, that although it may seem more fundamental to assign the two scaling factors, and hence to deduce the corresponding slip zone size which best fits the asymptote itself, this may not be easy to realise in a practical problem. This is because the value of K_{II} is defined by Eq. (13), i.e. it may be found from a consideration of the shear traction distribution very remote from the slip region: the difficulty is that, if the region over which collocation is attempted is too remote, the presence of other free surfaces in the component will probably have an influence on the traction distribution, so that the standard form associated with zero relative slip displacement will not be followed, i.e. it is impossible to define uniquely a region in which neither free surfaces nor the presence of the slip region has an influence on the

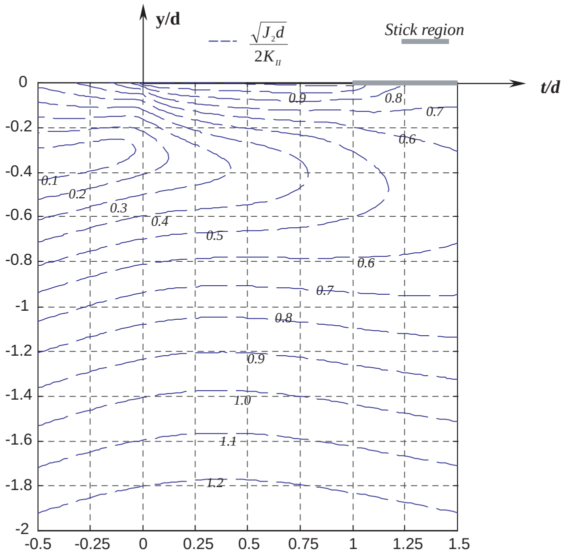

Fig. 2. Asymptotic stress field described by the von Mises parameter.

shear traction distribution. For this reason, it is expected that, in practical applications of the technique, it is K_I and d which will be chosen as the characterising parameters. In order to complete the picture the state of stress induced by the shear component of loading, given by Eqs. (16) is needed. This may be found from that for the sliding semi-infinite mode I asymptote, by a combination of superposition, shift and scaling, with the result

$$\Phi(w) = \frac{iK_{II}}{d}\left[\sqrt{w} - \sqrt{w-d}\right] \quad \forall w. \tag{19}$$

2.2. Properties of asymptote

The K_I asymptote has a very simple form, but the shear asymptote is a little more difficult to visualise and its universal characteristics are shown in Fig. 1(d). The ratio K_{II}/fK_I defines the size of the stick zone, d, from Eq. (18), but this may not be apparent from the figure because this is used as the non-dimensionalising length. Furthermore Fig. 2 displays the internal stress field as described by means of the asymptotes. Von Mises parameter has been chosen to describe the solution.

3. Example problems

We turn, now, to the application of these results to three example problems (Fig. 3). For each we will first derive the values of the scaling factors for each mode of loading. We will then compare the full stress field with that implied by the asymptote. It is too space-consuming to plot out all components of stress and so, for brevity, we will just display plots of von Mises contours, although we do not assert that this is necessarily the quantity responsible for crack nucleation. A second set of contours gives the discrepancy between the full field solution and the asymptotic field.

3.1. Hertz–Cattaneo problem

First, for the case of the Hertz–Cattaneo problem, Fig. 3(a), it is clear that, because the mode I asymptote itself was derived from these solutions, we may write down the scaling factor for direct loading as

$$K_{\mathrm{I}} = p_0\sqrt{\frac{2}{a}}. \tag{20}$$

The size of the stick zone in the Cattaneo contact is related to the direct and shearing forces applied (P, Q respectively) by

$$1 - \frac{d}{a} = \sqrt{1 - \frac{Q}{fP}}, \tag{21}$$

so that

$$\frac{K_{\mathrm{II}}}{fK_{\mathrm{I}}} = \frac{a}{2}\left[1 - \sqrt{1 - \frac{Q}{fP}}\right]. \tag{22}$$

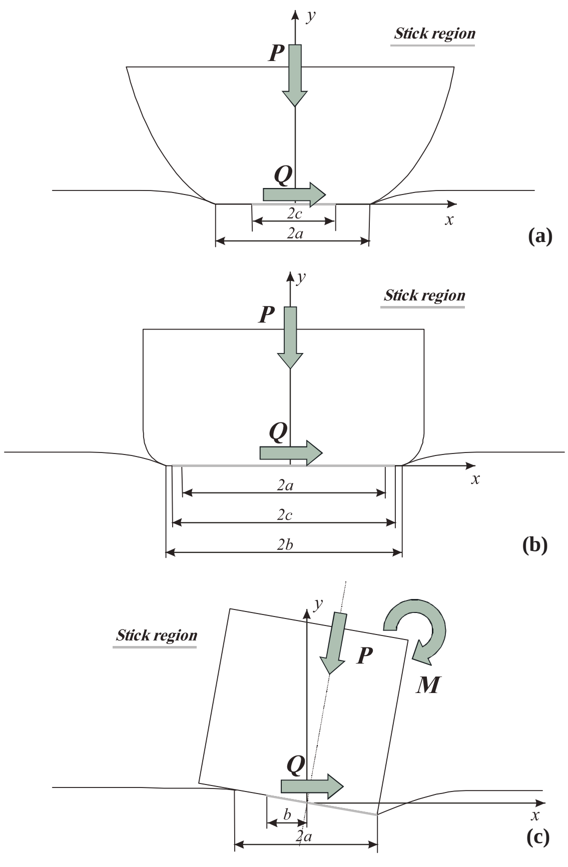

Fig. 3. Example partial-slip problems of incomplete contact problems: (a) Hertian contact, (b) flat and rounded and (c) receding tilted punch.

3.2. Flat and rounded punch

The geometry of the problem under consideration is shown in Fig. 3(b), and the cleanest form for the pressure distribution here is given by Dini et al. (2003) [1]

$$p(x) = \left(\frac{E^*}{2\pi R}\right)\left\{[2\sin^{-1}(a/b)-\pi]\sqrt{1-\left(\frac{x}{b}\right)^2}\right.$$
$$\left. - \ln\left(\left|\frac{\frac{x}{b}\sqrt{1-\left(\frac{a}{b}\right)^2}+\frac{a}{b}\sqrt{1-\left(\frac{x}{b}\right)^2}}{x\sqrt{1-\left(\frac{a}{b}\right)^2}-\frac{a}{b}\sqrt{1-\left(\frac{x}{b}\right)^2}}\right|^{x/b}\left|\frac{\sqrt{1-\left(\frac{x}{b}\right)^2}-\sqrt{1-\left(\frac{a}{b}\right)^2}}{\sqrt{1-\left(\frac{x}{b}\right)^2}+\sqrt{1-\left(\frac{a}{b}\right)^2}}\right|^{a/b}\right)\right\}, \tag{23}$$

where a is the half-length of the flat portion of the punch, b is the semi-width of the overall contact, R is the local edge-radius of the punch and E^* the equivalent Young's modulus for the contacting pair. If a shearing force Q $(< fP)$ is applied a standard partial-slip contact ensues, and the shearing traction distribution is given by

$$\begin{aligned} q(x) &= fp(x) \quad -b < x < -c \cap c < x < b \\ &= fp(x) + q^*(x) \quad |x| < c, \end{aligned} \tag{24}$$

where the corrective term can be obtained by invoking Ciavarella's theorem (Ciavarella, 1998)

$$q^*(x) = -\left(\frac{2fP\left(1-\frac{Q}{fP}\right)}{\pi c[\pi - 2\sin^{-1}(a/c) - \sin(2\sin^{-1}(a/c))]}\right)\left\{[2\sin^{-1}(a/c)-\pi]\sqrt{1-\left(\frac{x}{c}\right)^2}\right.$$
$$\left. - \ln\left(\left|\frac{\frac{x}{c}\sqrt{1-\left(\frac{a}{c}\right)^2}+\frac{a}{c}\sqrt{1-\left(\frac{x}{c}\right)^2}}{\frac{x}{c}\sqrt{1-\left(\frac{a}{c}\right)^2}-\frac{a}{c}\sqrt{1-\left(\frac{x}{c}\right)^2}}\right|^{x/c}\left|\frac{\sqrt{1-\left(\frac{x}{c}\right)^2}-\sqrt{1-\left(\frac{a}{c}\right)^2}}{\sqrt{1-\left(\frac{x}{c}\right)^2}+\sqrt{1-\left(\frac{a}{c}\right)^2}}\right|^{a/c}\right)\right\}. \tag{25}$$

Imposing equilibrium in the x direction between the shearing force and the shear-traction distribution yields the following explicit relationship between the applied loads and the stick zone size, c:

$$\frac{Q}{fP} = 1 - \left(\frac{c}{b}\right)^2\frac{\pi - 2\arcsin\left(\frac{a}{c}\right) - \sin\left[2\arcsin\left(\frac{a}{c}\right)\right]}{\pi - 2\arcsin\left(\frac{a}{b}\right) - \sin\left[2\arcsin\left(\frac{a}{b}\right)\right]}. \tag{26}$$

Lastly, we need the Muskhelishvili potential in order to determine the internal stress distribution. The full sliding solution has already been derived by Sackfield et al. (2002) and the closed form results for partial slip can be found by a combination of superposition, shift and scaling as follows

[1] Note that the neat form of the solution reported was derived from Ciavarella et al. (2002) correcting for typographical errors.

$$\Phi(w) = -\frac{E^*}{4\pi R}\left[\frac{1-\mathrm{i}f}{\mathrm{i}}\left(2\sqrt{(w/b)^2-1}\left(\pi/2-\sin^{-1}(a/b)\right)-\pi w/b-\mathrm{i}(w/b+a/b)T(-a/b,w/b)\right.\right.$$
$$\left.+\mathrm{i}(w/b-a/b)T(a/b,w/b)\right)+f\frac{c}{b}\left(2\sqrt{(w/c)^2-1}\left(\pi/2-\sin^{-1}(a/c)\right)-\pi w/c\right.$$
$$\left.\left.-\mathrm{i}(w/c+a/c)T(-a/c,w/c)+\mathrm{i}(w/c-a/c)T(a/c,w/c)\right)\right], \tag{27}$$

where

$$T(k,h) = \frac{1}{2}\ln\left[\frac{\mathrm{i}\sqrt{h^2-1}\sqrt{1-(k)^2}+(k)h-1}{\mathrm{i}\sqrt{h^2-1}\sqrt{1-(k)^2}-(k)h+1}\right]. \tag{28}$$

We are now in a position to determine the scaling factors. First, make the usual transformation of coordinates and place the origin at the edge of contact, so that the pressure distribution is given by

$$\begin{aligned} p(t) &= \frac{\sqrt{2}E^*[\pi-2\arcsin(a/b)]}{8\pi R(1-a/b)^{1/2}}\left[2\sqrt{td}+(t-d)\ln\left|\frac{\sqrt{d}-\sqrt{t}}{\sqrt{d}+\sqrt{t}}\right|\right] \quad t>0 \\ &= 0 \quad t<0. \end{aligned} \tag{29}$$

From this, we see that

$$K_{\mathrm{I}} = \frac{E^*}{2\pi R}\sqrt{2b}\left[\pi-2\sin^{-1}(a/b)\right]. \tag{30}$$

Turning to the effects of a shearing force, we require to replace the parameters b, c which effectively specify the size of the stick zone by d $(=b-c)$. Eq. (26) may be re-arranged in the required form, to give

$$\frac{Q}{fP} = 1-\left(1-\frac{d}{b}\right)^2\frac{\pi-2\sin^{-1}\left(\frac{a}{b-d}\right)-\sin\left[2\sin^{-1}\left(\frac{a}{b-d}\right)\right]}{\pi-2\sin^{-1}\left(\frac{a}{b}\right)-\sin\left[2\sin^{-1}\left(\frac{a}{b}\right)\right]} \tag{31}$$

and hence $K_{\mathrm{II}}/fK_{\mathrm{I}}$ is given by the solution of the following equation

$$\frac{Q}{fP} = 1-\left(1-\frac{2K_{\mathrm{II}}}{fK_{\mathrm{I}}b}\right)^2\frac{\pi-2\sin^{-1}\left(\frac{a}{b-\frac{2K_{\mathrm{II}}}{fK_{\mathrm{I}}}}\right)-\sin\left[2\sin^{-1}\left(\frac{a}{b-\frac{2K_{\mathrm{II}}}{fK_{\mathrm{I}}}}\right)\right]}{\pi-2\sin^{-1}\left(\frac{a}{b}\right)-\sin\left[2\sin^{-1}\left(\frac{a}{b}\right)\right]}. \tag{32}$$

3.3. Tilted punch (receding contact)

The last example we wish to consider is the titled punch depicted in Fig. 3(c). It is assumed that the magnitude of the overturning moment is sufficiently small to make the contact incomplete, and this problem has been briefly considered already by Sackfield et al. (2001) but additional results will be needed here. The pressure distribution is

$$p(x) = \frac{P}{a\pi}\sqrt{\frac{a-x}{a+x}} \tag{33}$$

and, in partial slip, the shear traction distribution is given by

$$\begin{aligned} q(x) &= fp(x) \quad -a < x < b \\ &= fp(x) - \frac{fP}{a\pi}\sqrt{\frac{b+x}{a-x}} \quad x > b \end{aligned} \tag{34}$$

with

$$b = a\left[1 - 2\left(1 - \frac{Q}{fP}\right)\right]. \tag{35}$$

Turning to the internal state of stress, the Muskhelishvili potential is given by

$$\Phi(w) = \frac{P}{2\pi a}\left[\frac{1 - \mathrm{i}f}{\mathrm{i}}\left(\sqrt{\frac{w+a}{w-a}} - 1\right) + f\left(\sqrt{\frac{w+b}{w-a}} - 1\right)\right] \quad \forall w \tag{36}$$

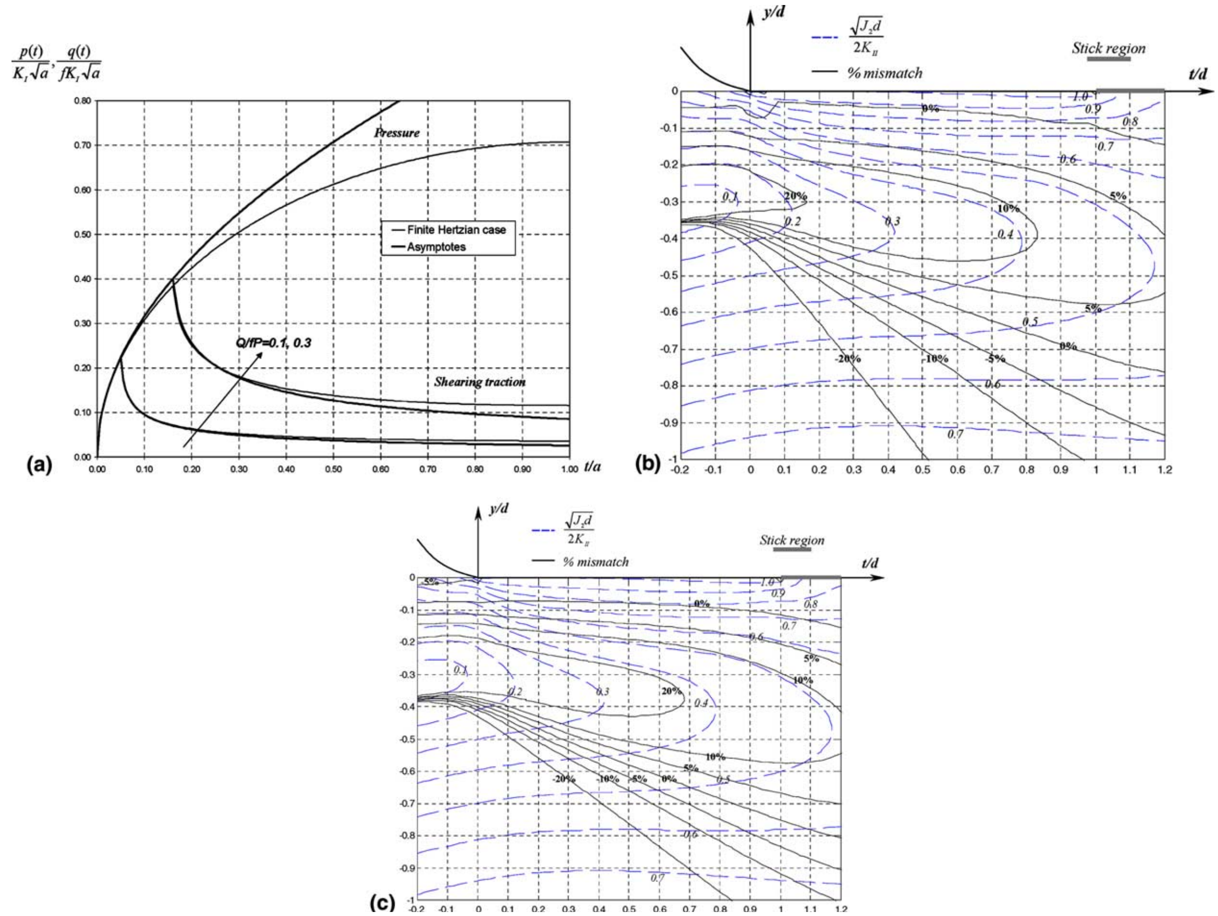

Fig. 4. (a) Normal and shear traction distribution: comparison between finite Hertzian contact ($Q/fP = 0.1, 0.3$) and bounded asymptote. (b) Contour plots of the von Mises yield parameter ($\sqrt{J_2}d/2K_{\mathrm{II}}$) and percentage mismatch between bounded asymptote and Hertzian contact characterised by $d/a = 0.051$, $Q/fP = 0.1$. (c) Contour plots of the von Mises yield parameter ($\sqrt{J_2}d/2K_{\mathrm{II}}$) and percentage mismatch between bounded asymptote and Hertzian contact characterised by $d/a = 0.163$, $Q/fP = 0.3$.

The same procedure as that followed above is used to match asymptotes: first, a change of coordinates may be employed to write down the contact pressure as

$$p(t) = \frac{P}{\pi\sqrt{2}a^{3/2}}\sqrt{t} \quad t \ll a \tag{37}$$

where t is measured from the contact edge, and from this, we see that

$$K_{\mathrm{I}} = \frac{P}{\pi\sqrt{2}a^{3/2}}. \tag{38}$$

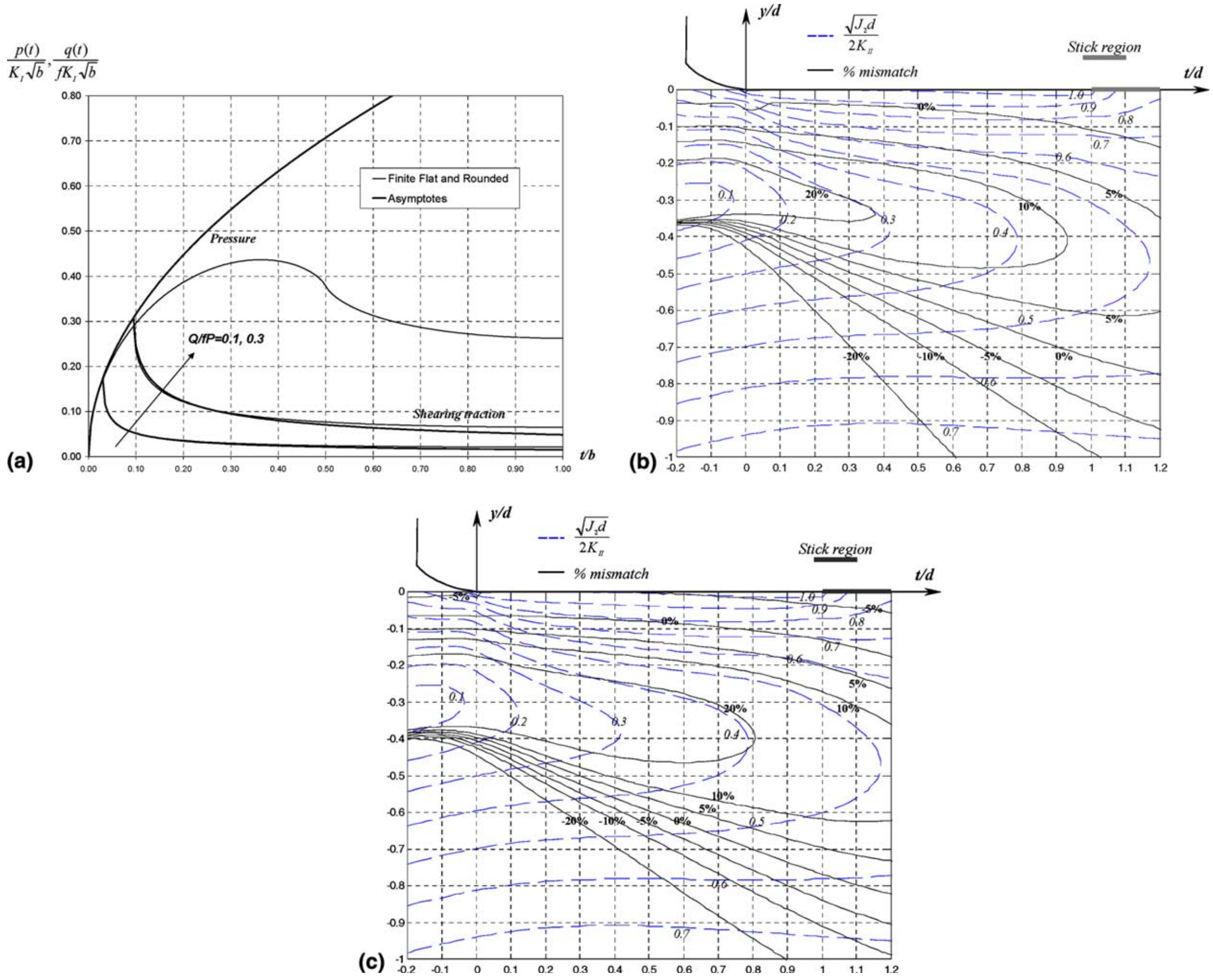

Fig. 5. (a) Normal and shear traction distribution: comparison between finite flat and rounded contact ($a/b = 0.5$, $Q/fP = 0.1, 0.3$) and bounded asymptote. (b) Contour plots of the von Mises yield parameter ($\sqrt{J_2 d}/2K_{\mathrm{II}}$) and percentage mismatch between bounded asymptote and flat and rounded contact characterised by $a/b = 0.5$, $d/b = 0.03$, $Q/fP = 0.1$. (c) Contour plots of the von Mises yield parameter ($\sqrt{J_2 d}/2K_{\mathrm{II}}$) and percentage mismatch between bounded asymptote and flat and rounded contact characterised by $a/b = 0.5$, $d/b = 0.095$, $Q/fP = 0.3$.

As before, we need an explicit equation for the size of the slip zone, d. From Eq. (35) we may write

$$d = a + b = 2a\left[1 - \left(1 - \frac{Q}{fP}\right)\right] \tag{39}$$

and hence arrive at

$$\frac{K_{\mathrm{II}}}{fK_{\mathrm{I}}} = a\frac{Q}{fP}. \tag{40}$$

4. Summary of comparison of contact fields

A few sample combinations of geometries and loading conditions have been considered for the three example problems. Figs. 4(a), 5(a) and 6(a) show the comparison between normal and shear traction

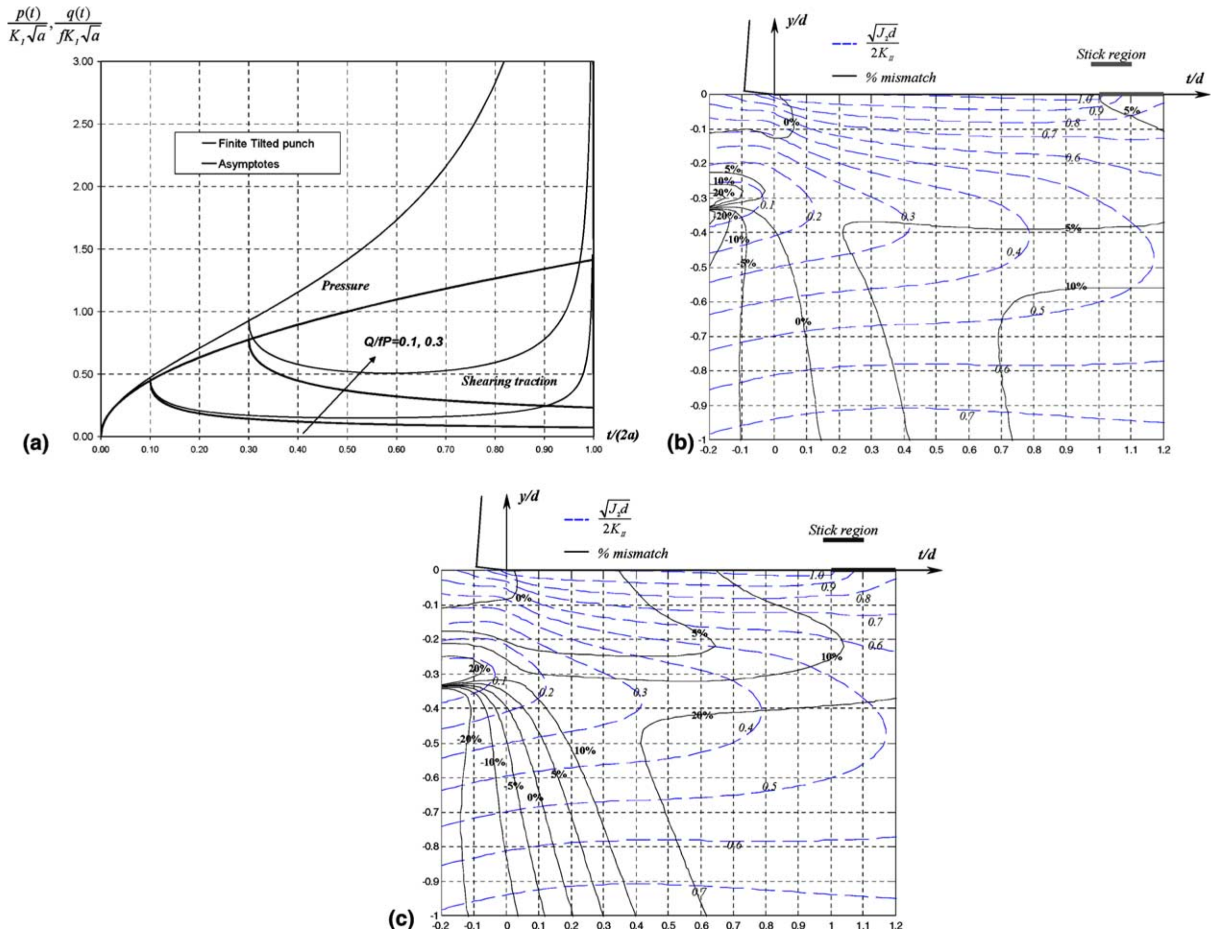

Fig. 6. (a) Normal and shear traction distribution: comparison between finite receding tilted punch ($Q/fP = 0.1$, 0.3) and bounded asymptote. Note that, in this case, the figure is drawn over a very extended region including all the contact width. (b) Contour plots of the von Mises yield parameter ($\sqrt{J_2}d/2K_{\mathrm{II}}$) and percentage mismatch between bounded asymptote and receding tilted punch characterised by $d/a = 0.2$, $Q/fP = 0.1$. (c) Contour plots of the von Mises yield parameter ($\sqrt{J_2}d/2K_{\mathrm{II}}$) and percentage mismatch between bounded asymptote and receding tilted punch characterised by $d/a = 0.6$, $Q/fP = 0.3$.

distributions between the asymptote and the finite contacts displayed in Fig. 3 in the same order. Similarly, Figs. 4(b,c), 5(b,c) and 6(b,c) display both stress state and fractional difference between the full solution and the asymptotic solution, again for the Hertz, flat and rounded punch and tilted punch problems respectively. The dotted contours represent the normalised von Mises yield parameter for the asymptote, $\sqrt{J_2 d}/2K_{II}\left(\equiv \sqrt{J_2}/fK_I\sqrt{d}\right)$, whilst the solid curves describe the discrepancy between the respective von Mises' yield parameters in terms of percentage mismatch $\left(100\left(\sqrt{J_{2\text{finite}}} - \sqrt{J_{2\text{asymptote}}}\right)\left(\sqrt{J_{2\text{finite}}}\right)\right)$: von Mises' parameter is used simply as a single parameter measure of some qualities of the state of stress, and it may well be that the crack nucleation criterion depends on some other stress parameter. Clearly, different components of stress will diverge at different rates as the observation point moves away from the contact corner.

It is clear that, in all cases, there is very good agreement between the full field and the asymptote along a narrow band close to the surface. The regions plotted display a cross section of approximately $2d \times 2d$ which is the region of interest, although it may be that the actual region in which relevant irreversibilities occur is rather smaller. It is clear that, down to a depth of $0.2 \times d$ the asymptote and full field usually agree to within 10% of each other. The convergence of the two solutions is, in fact, rather better than this in a few cases (see Fig. 6(b) for the tilted punch). Also, this is a very much tighter tolerance on the mismatch between the solutions than one normally imposes in a fracture mechanics calculation, where small scale yielding conditions may lead to mismatches of 40% based on the same parameter.

5. Conclusion

A method for encapsulating the complete stress field in the neighbourhood of crack nucleation in incomplete contacts suffering partial slip, using only two parameters, has been derived. This provides a means of correlating the fretting fatigue performance, whether in terms of the infinite-life/finite-life boundary, or, if the latter, the number of cycles to nucleate a crack. The sole requirement is that the process zone which characterises the region within which non-linear behaviour gives rise to irreversibilities eventually leading to nucleation be contained well within a region in which the stress state is well characterised by these parameters.

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